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Quanta Perceived as Quaternions

Contents

Abstract

Quaternions are spherical particles of three-dimensional space and possess two complementary systems of coordinates. A quaternionic sphere, a conformal figure without centre, diameter and locality may be seen both as a material point and as an element of a wave. Physical movements can be described as motions of quaternionic spherical particles. In using linear vector coordinates a moved spherical quaternionic quantum may be seen as the motion of an elementary physical particle located at a point. Moved quaternionic quanta described by circular angle coordinates appear as elements of a wave. In seeing numbers of the Hamiltonian skew field as first elements of our natural space, physics possesses one new comprehensive model -- free of contradictions — that comprises the classical duality of waves and particles.

1. Introduction

E. Schrödinger was always convinced that physics is been lacking a total and complete picture of both the particle and the wave aspects of every (micro)physical movement [27]. R. Penrose and other scientists are looking for a form of quantum mechanics that can for instance better interpret 'non-local', 'paradox' EPR-phenomena [20].

Quaternions[1] can be perceived as elements of the 3-dimensional space of our visual perception and of our physical experience [22,25]. I have exposed in [26] that these quaternionic particles can be trigonometrically described in a twofold manner. Therefore these spherical quaternionic elements possess a system of complementary coordinates. So I am guided to the hypothesis that Schrödinger's complete picture of moving material particles can be achieved, if the elements of the Hamiltonian skew field **H** are perceived as the quanta of mechanics. Using this way also non-local phenomena get a natural understanding.

A critical reviewer may wonder that still quaternions are interesting aims of scientific investigations. Already at the beginning of last century more than thousand articles were published, which touch quaternionic topics (Bibliography of A. Macfarlane, Quaternions and allied systems of Mathematics, Dublin 1904, 86 pages). But for example 1996 S. de Leo could collect 15 articles published in the decade 1985–95 in physical journals, discussing the physical use of quaternions (Cf. Reference 2 in [15]). In [22], Section 2, I refer to four attempts to construct a quaternionic quantum system [1,10,29,31]. These papers and other approaches (as [2,15,16,19,30]) are not comparable with my identification of quanta and quaternions. Mostly these articles use 'complex' quaternions, not the original Hamiltonian skew field numbers. For instance J. D. Edmonds [10] tried to construct a "relativistic quantum theory of the ring of complex quaternions". Only de Leo emphasized, "that a complexfied quaternionic version of Special Relativity is a choice and not a necessity" [15][2].

Dirac's applications of quaternions to Lorentz tansformations [7] are especially noteworthy. J.L. Synge shows the connection between the quaternionic form of Lorentz transformation and Dirac's algebra [30]. Davies, Finkelstein, Jauch, McKellar and Speiser study interesting special aspects of quaternionic quantum mechanics [5,6,11][3].

[1]The websites of E. W. Weisstein http://mathworld.wolfram.com/Quaternion.html may be read by students who do not know the classical concept of quaternions.

[2]Also in his website http://quaternions.com "Doing physics with quaternions" D.B. Sweetser uses 'real' quaternions for describing Lorentz transformations.

[3]In [17,18] S. de Leo — together with G. Ducati and C.C. Nishi — analyses the quaternionic tunnelling effect with some interesting consequence in DP violating quantum systems.

But these previous articles do not use a conformal triangle model of quaternions in three dimensions and the complementary coordinates of quaternions. Quaternionic numbers cannot only perceived geometrically as 'points' and 'vectors' but also as conformal 'triangles (trigons)' and as conformal 'spherical particles [22, 25]. This finding uncovers the complementary, double 'triangle' and 'vector' aspects of mathematical numbers. In essence these complementary pictures (both vector and triangle) reveals geometrically the two complementary physical aspects of a quaternionic quanta (both 'particle' and 'wave')[4].

2. Special Relativity in Linear Coordinates

Basic equations of special relativity can be formulated with the help of quaternions and their linear coordinates. This formulation is a compact, formally new one ([26], Sections 6,7; physical part of [22]). But this formulation also makes possible a new view of special relativity.

Though Einstein took over from Minkowski the 4-dimensional description of relativity, he always steered clear of non-real numbers by describing his theories. Einstein (and many other physicists) could see the usage of an imaginary time, proposed by Minkowski, as only a formal trick to facilitate reckoning. My perspective is an antithetical one: Schrödinger perceived not only real numbers (or vectors with real components) but also complex numbers ('Ψ-numbers') as physical entities[5] . This raises the question whether some physical structures described by special relativity may also be seen more fundamentally if complex and quaternionic numbers are used for expressing these structures.

Indeed the physical second part of [22] and Sections 6,7 of [26] exemplarily illustrate how

[4]Often vector algebras, remnants of quaternionic algebra, cannot reflect the conformal aspects of numbers. A tetraglobic triangle, the conformal picture of a number, generally possesses a cross ratio (double ratio) as metrical number. Only in special ("Euclidean") locations (Definition 2.3, [24]) this metrical number can be seen as a simple ratio (quotient) of two vectors (in the sense of [13,14]). In using length-metrical concepts of elementary vector calculus I have proved in [22] that the multiplication of quaternions can be seen as the composition of Euclidean triangles. (Naturally, this proof can be tightened in using 'higher' vector formalisms, for instance multivector algebras). But essentially my conformal picture of quaternionic numbers is not a vector picture but a tetraglobic triangle picture [24,25].

[5]At least in the twenties, in the phase of discovery, Schrödinger had nothing to do with this concept. Still in his fourth paper "Quantisierung als Eigenwertproblem", 1926, he said: "Eine gewisse Härte liegt ohne Zweifel zurzeit noch in der Verwendung einer komplexen Wellenfunktion". Still in this phase the idea of generally and absolutely complex values of the Ψ-function was not obvious for the discoverer of this function ([28], p.171). The physical meaning of this function is still disputed and debatable but surely and generally the Ψ-numbers are complex numbers.

this question can be discussed. This way Lorentz-transformations could be seen as a special description of an elementary function $z \to z' = z.W$ on **H**. This function describes not only a coordinate transformation between two systems of observers. The mathematical mapping $z \to z'$ between quaternionic numbers can also physically seen as a movement of particles in the natural 3-dimensional space. $z \to z'$ describes the elementary movement[6] of a quantum z into a quantum z'.

3. Connection to Schrödinger's Mechanics

Quanta are conformal spherical particles in the natural 3-dimensional space. Particles z, elements of **H**, can be described in a twofold, complementary manner. Schrödinger has used the traditional (circular) coordinate representation of complex numbers in describing elementary quantum mechanical movements. The quaternionic form of special relativity formulated in linear (hyperbolic) coordinates is associated to Schrödinger's quantum mechanics in a complementary manner. Both theories use *formally* the algebra of quaternionic complex numbers. Both theories describe the movements of quanta. Quanta *are* quaternionic complex numbers.

Schrödinger's form of quantum mechanics at first only gets along the common complex numbers, that means along special quaternions because often, if for example a simple atomic structure is calculated, the spatial position of a quantum[7], described by the unit vector **i**, is not important. Therefore in these situations the quaternionic **i** can be identified with the common imaginary i. But if not the wave but the particle form of a movement has to be described, the direction **1** of this movement is always important, so that for shaping this structure of movements the non-commutative, quaternionic form of complex numbers is often a necessary form.

4. Imaginary Metrical Numbers

To what extent can a newly perceived quaternionic special relativity be identified with Einstein's theory, where different structures are observable? What importance has the raising of *imaginary units* for describing physical *metrical units*?

In this connection one has to remember that Minkowski attributed the imaginary unit i to the parameter of time, in contrast to the situation when *linear coordinates* are used physically. In linear coordinate systems [26] a vector coordinate $^\wedge a$ has an imaginary metrical unit. Certainly both systems, the circular and the linear coordinates, are equivalent mathematically. The two complementary coordinate systems realize mathematically only a twofold, isomorphic description of the same formal structure **H**, its elements, its algebra and the trigonometry of its numbers. (I cannot see a mathematical argument supporting that a real number as a metrical number of — for example — the right angle, is more natural than an imaginary metrical number: Imaginary and real angle units are equivalent because angle measures ($^\wedge\varphi$ or $\varphi^\wedge$) are only added. A difference of these units could only be realized by multiplication)

One question is the mathematical equivalence; a different matter is the diverse physical meaning of both complementary representations. Can only geometrical angles be measured with an imaginary unit? Can physical vectors still be seen as physical vectors if they possess an imaginary metrical unit?

5. The Physical Space Seen as Imaginary Axis

If the quaternionic version of special relativity is accepted the Cartesian, the point-concept using description of a movement is more natural by using not a real but an imaginary unit for describing the three Cartesian coordinates of this point. The 3-dimensional world of our experience quasi becomes the (3-dimensional) axis of a Gauss/Argand plane (Cf. Section 5 in [25]). Einstein's intuition contended (as far as this is concerned in opposition to Minkowski) that the assumption $t^2 \leq 0$ for the *time parameter*[8] t is a non-physical assumption. Only the assumption $x^2 \leq 0$ for the parameter of location x leads to the "*Cartesian and Pythagorean*" identification of numbers and quanta postulated by a quaternionic skew field form of theoretical mechanics.

The quadratic norm $Q.Q^*$ of a quaternion Q is traditionally only be represented 'positive definitely' whereas the basic quadratic form, seen as fundamental by Einstein, is not 'positive definite'. This ostensible contradiction disappears with the help of quaternionic complementary coordinates: I make a distinction between the representation of quaternions by their components, $Q = t + x$, without using coordinates, so that

$$Q.Q^* = (t + x).(t - x) = t^2 - x^2 \qquad (5.1)$$

[6]The concept 'physical movement $z \to z'$ in the 3-dimensional space of our perception' also makes sense if a mathematical reduction of the numbers z, z' in their kinematical components (t, x) or their dynamical components (m, p) is (physically) without sense. Cf. [26], Section 8.

[7]The question if for example different directions $\mathbf{i}_1$, $\mathbf{i}_2$ of two quanta can be identified with the spins of two atomic electrons is still not discussed.

[8]Klaus Habetha has a false understanding of the quaternionic theory if in his review of my article [22] (Zentralblatt MATH 2001) he commented, "In his system there is no difficulty to deal with real time as well as with complex time." In my system time is always real. Under some circumstances the using of a (real) metrical unit of time loses its physical sense.

is not 'positive definite'; and the representations $Q = t + \mathbf{1}^\wedge x$ and $Q = t + \mathbf{i}x^\wedge$ with the help of the complementary *coordinates* [26] $^\wedge x$ and $x^\wedge$ so that

$$Q.Q^* = (t + \mathbf{1}^\wedge x).(t - \mathbf{1}^\wedge x) = t^2 - (^\wedge x)^2$$

or (5.2)

$$Q.Q^* = (t + \mathbf{i}^\wedge x).(t - \mathbf{i}x^\wedge) = t^2 + (x^\wedge)^2.$$

Therefore the quadratic norm $Q.Q^*$ is positive definite only in circular coordinates. By remaining in the quaternionic calculus (without going back to coordinates) or in using linear coordinates the typical Einsteinian minus-sign[9] appears in the quadratic norm $Q.Q^*$.

That the traditional quadratic form is not positive definite is based on the metrical axiom $0 \leq x^2$ of Einstein's strictly real version. In the structure of quaternions a relativistic quadratic form gets its place if the axiom $0 \leq x^2$ is substituted by the axiom $x^2 \leq 0$. This substitution is realized if the traditional unit $h = 1$ of the coordinates x is replaced by the unit $h = i$ of the linear coordinates $^\wedge x$[10].

6. The Non-periodical Aspect

In writing complex Ψ-numbers with the help of the usual circular system Schrödinger uses — as a matter of course — the periodical aspect ('wave aspect') of quaternionic numbers. By writing these numbers in using the linear system a 'non-periodical' aspect of these quaternions (particles, quanta) catch one's eye if the hyperbolic form

$$W :=^\wedge W = \exp(\mathbf{1}.^\wedge\varphi) = \cosh^\wedge\varphi + \mathbf{1}.\sinh^\wedge\varphi \quad (6.1)$$

([26], (5.1)) is used. But also together with a relativistic, *real* $^\wedge\varphi$ the linear vector coordinate $^\wedge x$ is always *imaginary*[11].

I see two ways to understand physically this mathematical situation:

- Either the quaternionic calculus grants only a very compact style of writing special relativity. If somebody wants to migrate from this compact calculus to the 'real' situation the 'fictitious' non-real coordinates of the linear system have to be substituted by coordinates with real units.
- Or the quaternionic modification of special relativity is more reasonable, formally and physically. For this way the wave and particle aspect of physical movements can be seen without contradictions, realized in one model of physical quanta. The relativity part especially describes the corpuscular aspect. But also in this part the periodical aspect of every movement is *implicitly* conserved.

I am an advocate of the second point of view.

7. On Historical Developments of Measurements

Modern mathematics measures the diagonal of a unit square by $\sqrt{2}$. Greek mathematicians would not accept this 'irrational' number as a metrical number. Can modern physics accept $i = \sqrt{-1}$ as a metrical number?

We should remember that the angle concept and its measurement have passed through several steps: Euclid did not know a connection between the measurement of angles and the measurement of lengths (for example on a unit circle). Euler has taught us to regard the transcendental number π as a natural metrical number for measuring the sum of angles in Euclidean triangles. Because the trigonometry of complex numbers can be described in two complementary systems [26], science should also accept πi as a natural metrical number.

Mathematics and physics did not restrict themselves to the concept of trigonometric angles. The concept 'sine-curve' is first known after accepting any real number as an argument of trigonometric functions. The periodicity of $\sin\varphi$ and the non-periodicity of $\sinh\varphi$ are involved only together with this non-trigonometric view of angles.

With good reasons Euler introduced number π as a natural metrical unit of both trigonometric and non-trigonometric angles. But until today practical mathematics uses mostly 360 degrees, not 2π radians, for measuring a complete turn. Physics can accept $2\pi i$ analogically as a natural measure of angles if corpuscular movements are described theoretically in such a form that this 'non-periodical' form of particles has also maintained *implicitly* the complementary, periodical aspect. Classical real units are adequate to describe

[9] The existence of complementary quantum coordinates also explains the seemingly paradox side by side of definite and non-definite dynamical quadratic forms in relativity and quantum mechanics: The concept of rest mass is based on the relativistic, non-definitive form (e.g. V. Fock [12], equation (28.11)). In founding his quantum mechanics Dirac used a positive quadratic form (cf. [8], §30, (23) and §67, (3)).

[10] Section 3 in [26] emphasized that the symbol **i** can mark two different objects, both a particle (i.e. a quaternionic number) and the position (direction) of this particle. Accordingly I see a difference of the number i and the 'denomination' $\mathbf{i}$.

[11] As is well known also Einstein's 'strict real' version can be formally written in using hyperbolic functions with real arguments (i.e. equation (17.32) in V. Fock [12], and Dirac, [8], §67, (19)). Therefore the quaternionic form of mechanics presupposes that sometimes imaginary arguments of hyperbolic functions are more reasonable physically than real arguments. This supposition only seems paradox if 'imaginary' is used synonymously to 'physically non-real'. A quaternionic mechanics has to accept both 'imaginary' and 'real' numbers as representatives of mechanical realities.

a ('purely non-periodical') movement of points in some areas, for instance in macrophysics. Here one can almost forget that the linear coordinates $^\wedge x$ possess an imaginary unit in a generally theoretical context. And vice versa: The strictly real formalism of special relativity is 'formally flawless'. But '*this formalism still does not know*' that its vectors $^\wedge x$, $^\wedge v$ possess an imaginary unit — in reality. The term 'in reality' means both 'in relation to an optimal coordinate system' and 'in a physical space possessing quanta (quaternions), not points as first elements'.

8. The Conformal Duality of Tetraglobic Quaternions

Quaternionic foundation of wave and corpuscle mechanics is based on the conformal trigon-representation of quaternions. Trigons are centred and directed tetraglobes [24,25]. The foundation of conformal trigonometry of tetraglobes leads to a concept of angles that does not need the idea of straight lines (Definition 6.2 in [23]). All conformal basic figures, for example 2- and 4-circles, are dual in points and conformal circles. I perceive conformal points as a geometrical substratum of corpuscular movements, conformal circles as basic elements of waves. The dual structure of conformal basic figures still reflects geometrically the possibility of a twofold, complementary coordinate description of quaternionic numbers. And this complementary structure of the numbers expresses mathematically that the duality of all physical movements is rooted in the complementary structure of physical quanta. 'Numbers', 'quaternions', 'trigons', 'spherical particles' and 'quanta' turn out to be synonyms.

Only if one point of the coordinate-tetraglobe is especially distinguished as the absolute Euclidean point (Definition 4.1 in [25]) the conformal dual symmetry of points and conformal circles is destroyed. Only if this 'point of infinity' exists the one set of conformal circles disintegrates into the two sets of Euclidean straight lines and of Euclidean circles. The corpuscular theory of physical movements ('relativity') is based on the inertial movements along Euclidean straight lines. This theory seized an essential aspect of physical movements. But this does not change the fact that a Euclidean straight line, seen from the higher conformal standpoint, is a conformal circle, which always possesses the topological connection of a circle. The description of the relativistic aspect with the help of quaternionic linear coordinates conserves implicitly this periodical structure of all circles. But this conformal aspect cannot be seen if the simplest corpuscular movement is only interpreted as the constant inertial movement of a (mass)point along a Euclidean straight line.

A general theory of physical movements which accepts that geometrical angles and physical movements are similar entities, which accepts that angles can be defined without using the concept 'straight line', should also realize that physical movements cannot always be described with the help of 'straight lines'. But special and general relativity in their traditional forms always require this axiom.

9. Aspect Free of Coordinates

Skew field **H** possesses a level whose mathematical description in principle does not need the two complementary coordinate systems. I call this level the elementary geometrical, algebraic trigonometric or purely algebraic aspect and imagine that this algebraic area does not touch the physical dualism.

Above I expect that fundamental physical principles (of both micro- and macrophysics) can be described primarily with the help of (as) functions[12] on **H**. Then these functions can be understood as the primary 'complex' (still free of coordinates) expressions of physical movements[13], which only appear circularly or linearly depending upon the situation of an observer (the arrangement of an experiment). A material quantum seen with the help of a circular coordinate system (characterized by $h = 1$) appears as the element of a wave. A material quantum seen with the help of a linear coordinate system (characterized by $h = i$) appears as a point.

10. Physical Space without Measurements of Lengths

Lorentz-transformations in their kinematical (space-time-like) or their dynamical forms describe especially the function $z \to z' = z.W$ on **H** with the help of linear coordinates $x = \mathbf{1}.^\wedge x$ or $p = \mathbf{1}.^\wedge p$. In these forms the Lorentz-transformations are only the expressions of elementary corpuscular movements of the quaternionic particles. But the mover W also contains explicitly a periodical structure, which is realized in circular coordinates and can be observed as a wave. Schrödinger has used these classical circular coordinates to describe the Ψ-numbers of his quantum mechanics. The quaternionic model of quanta produces new possibilities to give Schrödinger's

[12] Theorems of complex analysis often do not relate to coordinate representations of complex numbers. If my conception proves its worth that first elements of physical space are quaternionic quanta, the world of our experiences has to be seen as a *projective number line* **H** whose points are quaternions. Then theoretical physics has to inquire the physical meaning of the complex function theory on **H**. This theoretical science has to ask especially which parts of this function theory can be understood as theoretical mechanics.

[13] $z \to z' = z.W$ is a function that free of coordinates describes the simplest movement of a quantum. Newton's constant movement $t \to x' = t.v$ may be seen as the classical analogue.

Ψ-numbers a physical meaning.

In [26] the physical meaning of $z \to z' = z.W$ written in circular coordinates was not discussed. In the physical part of [22] this basic transformation was written in its components (and in its kinematical and dynamical forms) but here also this transformation was only interpreted by means of Einstein's space-time-scheme. Generally, a periodical movement of material quanta written in circular coordinates ('movement of a wave') must not be seen as a real 'movement that is located along a (3-dimensional, space-like) x-axis'. Naturally also a circular material quantum is moving in the 3-dimensional space of our experiences. But a compulsion does not exist to underlay this kind of movement with the classical space-time-scheme of Newton and Einstein, which is especially established by length-metrical structures. It will often be sufficient to think of the 3-dimensional physical space as a purely conformal one, only produced by conformal quaternionic tetraglobes. In this conformal space the splitting of a quantum into its real and vector parts (either $t + x$ or $m + p$) will often lose[14] its physical sense. At first only the measurement of angles is necessary, not the measurement of 'lengths' to determine this kind of particle.

$z \to z' = z.W$ should not only be designated as 'Lorentz-transformation' (i.e. as a coordinate transformation in the frame of Einstein's space-time-structure) if my complementary interpretation proves its worth. A new denomination as 'Euler- or Einstein-transformation' would be more convenient. If the trigons z and W possess different directions, also $z.W \neq W.z$ must be taken into account[15].

11. The Conformal View of Velocities

Quantum mechanics and mechanics of special relativity can be still more distinctly seen as a unit if the different roles of the numbers W and v in both theories are compared. Relating to this it is helpful that both numbers at first can be understood as pure invariants of conformal geometry, without references to circular or linear coordinates [26].

The representation of v and W in linear coordinates is characteristic for the corpuscular, relativistic aspect where v and W are velocities in the traditional sense. In Schrödinger's mechanics a wave is described in the form $r.W^\wedge = \rho.\exp(\mathrm{i}.\varphi^\wedge)$. In circular coordinates W is represented by $W := W^\wedge$. The mathematical possibility to write W as the sum of real and vector parts is often a non-essential one in this circular area. In both theories W and v are 'mover' (movement numbers)[16] but they appear as 'velocities' (in the traditional sense as ratios $v := x/t$, $W := Q/\tau$) only in the space-time-scheme. Generally, on the conformal level W and v have to be seen as cross ratios (double ratios)[17]. Section 13 will discuss this in more details. Einstein's special relativity promoted the fundamental role of the velocity of light. Section 13 will show that all 'velocities' possess a fundamental role because they can be seen — before any length-metrical interpretation — as invariables of conformal angles. Naturally in its traditional understanding 'velocity' only has a physical sense in relation to a (Cartesian) system of an observer. But 'velocity' seen as the characteristic number ('mover') of a quantum is associated with this spherical particle independently of a Cartesian system of coordinates. In this sense the conformal invariance of W and v is comparable with the invariability of the velocity of light if the observer is changed.

12. Metrical Units in Their Coordinate Systems

Any station of measurement that a physicist can use is a macro-physical station. Any observer is sitting in a macro-physical coordinate system.

For any description of physical macro-phenomena a physicist possesses a coordinate system that in particular allows the measurement of angles, times and energies. In this area of experience simultaneous measurements of times and energies are also possible.

For a description of physical micro-phenomena the above units of measurements are only available in a restricted extent. In observing a micro-physical movement an exact simultaneously measurement of its time and energy is not possible (Heisenberg). A physicist may prognosticate statistical features of a movement but if he wants to measure he has to decide whether he wants to measure the kinematical or the dynamical aspect of this movement.

Does an extreme microphysical area exist, which can be grasped exactly only with the help of angle-quantities, that means by cross ratios and trigon-functions? In principle the description of quaternionic movements with the help of Schrödinger's quantum

[14] or only gets a physical sense again if *the observer compelled the localisation* of a quantum by introducing a length-metrical Cartesian system.

[15] The discussion of $z.W - W.z$ may show that the quaternionic difference $Q.P - P.Q = x.p - p.x$ (of kinematic and dynamic events $Q = t+x$ and $P = m+p$) is related to Heisenberg's difference in his matrix equation $Q.P - P.Q = hi$. In this case Heisenberg's hi can be vividly interpreted: Real physical movements are quantized; nature only permits quantifications by complete turns. Cf. Section 16 that interprets hi as a natural unit of angle measurement.

[16] Here I see a touch to the attempts of L. de Broglie to get not just a statistical interpretation of the Ψ-function [4]. But de Broglie could still not realize that complex numbers (and therefore also velocities) can be seen as conformal entities.

[17] In the common 'strictly real' version of special relativity also a double ratio comes into the open already when velocities — for instance in the Lorentz-transformations — appear only in the form v/c of a double ratio.

mechanics is already restricted to this conformal area of angle-quantities, if the additive representation of a number by its sum of real and vector part has lost its physical sense. For example this sense is lost if the separation of a movement in its time and its (length-metrically determined) space components has lost its sense.

Only its angle coordinates and its direction (**i** respectively **1**) determine a quantum in its conformal form; only the measuring of angles is necessary to determine physically this number on the conformal level.

If an observer has decided to describe a movement in its corpuscular form he can use his Cartesian coordinate system (which also possesses length-metrical units) to realize a bijection between the moving quantum and a geometrical point. On this way the quantum becomes a pointer to this (mass)point. Therefore in this special situation the bijection of points and quanta leads back to Newton's concept of a moved mass point (cf. [25], Sections 4,5)[18].

13. Conformal and Length-metrical Interpretation of Numbers

Generally quaternionic numbers possess geometrically a spherical shape. Numbers, seen as spherical particles are purely conformal figures. The four points and the four conformal circles of these figures (Tetraglobes, Definition 2.1, [24]; Trigons, Section 1, [25]) are well defined in their mutual positions by the three triangle angles possessing the Euclidean angle sum. But these spheres do not have a diameter. The geometrical extent of these particles is not defined.

Here we get — on an elementary geometrical level — a 'non-local' situation. The only foundation for a conformal metrical destination of these particles is the measurement of angles. Article [26] developed several metrical parameters of a conformal angle, namely the numbers v, W,

$$W = \gamma(1+v) = \gamma + \gamma.v, \ \gamma := (1-v^2)^{-1/2}, \quad (13.1)$$

$$v = (W - W^*)/(W + W^*) \quad (13.2)$$

(cf. (2.5), (2.4) in [26]). v and W are connected with $^\wedge\varphi$, $\varphi^\wedge$ by

$$v = T(\varphi) = \mathbf{i}.\tan\varphi^\wedge = \mathbf{1}.\tanh{}^\wedge\varphi, \quad (13.3)$$

$$W = C(\varphi) + \mathbf{S}(\varphi) = \exp \mathbf{i}\varphi^\wedge = \exp \mathbf{1}{}^\wedge\varphi \quad (13.4)$$

(cf. (2.6) and Theorem 5.1 in [26]). An angle φ is measured indirectly by v and W with the help of cross ratios, the characteristic numbers of special triangles [26]. An angle φ is measured directly by the complementary metrical numbers $^\wedge\varphi$ and $\varphi^\wedge$. This measurement is a direct one because the unit employed is the unit angle ('complete turn') measured by $2\pi h$ ($h = i$ or $h = 1$). These measurements of angles are natural because the addition of two angles φ_1, φ_2 and the addition of their measures $^\wedge\varphi_1$, $^\wedge\varphi_2$ (or $\varphi_1^\wedge$, $\varphi_2^\wedge$) always mutually correspond (cf. [26], Section 7).

Also ρ, the norm of a quantum, is a conformal invariant. But the discussion of the conformal sine law in [24], Section 9, has shown that 'ratio of sides', which in principle can be defined, on the conformal level must not be seen as the quotient of two length-metrical quantities.

In [22] a quaternion $A = \rho.W = \alpha + \mathbf{a}$ was interpreted physically as follows: If the real part describes the time t and the vector part the location $\mathbf{x}$ of an event then this leads to the kinematic interpretation Q of a quaternion

$$Q = (t, \mathbf{x}) = t + \mathbf{x}. \quad (13.5)$$

In viewing the real part as mass m and the vector part as impulse $\mathbf{p}$ the dynamic interpretation

$$P = (m, \mathbf{p}) = m + \mathbf{p} \quad (13.6)$$

of a quaternion is obtained.

From that it follows that the norm ρ of a quaternion is the proper time τ if this quanta is a kinematical event. The norm ρ is the rest mass μ if this quantum is a dynamical event. The velocity v can be seen as ratio $v = \mathbf{x}/t$ or $v = \mathbf{p}/m$. And the quaternionic velocities ('4-velocities') $W = Q/\tau$ or $W = P/\mu$ are associated with these vector velocities v (cf. [26], (2.5)).

The triangle model of quaternions developed in [22] uses the traditional length-metrical measurements. The description of a position $\mathbf{x}$ with the help of a (Cartesian) coordinate system requires the measurement of lengths. In a figurative sense also the magnitudes $\mathbf{p}$, m, t are 'length-metrical' concepts.

How does the especially length-metrical interpretations (13.5) and (13.6) come into being if a quantum is generally only a purely conformal geometrical/physical basic figure?

If this general conformal situation is the starting point the kinematic and dynamic description of a quantum in the sense of (13.5) and (13.6) is only possible with special assumptions as

- a clock exists whose time unit can be used to fix the proper time τ of the quaternion, or
- a measuring instrument and a metrical unit exist so that the quaternionic norm can be described by the rest mass μ.

With these preconditions and with (13.1) it follows that for a kinematic event

$$Q = \tau.W = \tau.\gamma(1+v) = \tau\gamma + \tau\gamma.v, \quad (13.7)$$

[18] Section 8 of [24] explains that a tetraglobe (and therefore also the spherical particle of a trigon) can geometrically picture a physical quantum either as a point ('corpuscle') or as a plane ('wave element') because a diameter of a tetraglobe is not defined.

and for a dynamic event

$$P = \mu.W = \mu.\gamma(1+v) = \mu\gamma + \mu\gamma.v. \qquad (13.8)$$

With the definitions of scalars t, m by

$$t := \tau\gamma, \quad m := \mu\gamma, \qquad (13.9)$$

and of vectors x, p by

$$x := t.v, \quad p := m.v, \qquad (13.10)$$

we return to the kinematic and dynamical interpretations (13.5) and (13.6). Therefore by starting on the conformal level the position vector x and the impulse vector p are defined with the help of v. On this level v is only a cross-ratio that is (indirectly) measuring a conformal angle φ. Physically I see this angle as a 'conformal frozen' form of a movement. Certainly the two vector equations (13.10) can also formally be written inversely:

$$v = x/t, \qquad v = p/m. \qquad (13.11)$$

This means that the classical concept of v as quotient of two length-metrical quantities is completely reproduced if x and p can be measured as lengths, independently of the conformal definitions (13.10). One should note that the equations (13.9) describe the usual relativistic connection between time and proper time (and between mass and rest mass) *after* this traditional understanding of velocities is reproduced.

The conformal structure of quaternions is the base (the background) of the classical length-metrical description of physical movements. Classical mechanics only understands v, W as quotients of two length-metrical quantities, not as measuring numbers of angles = frozen movements. These conformal 'movers' v, W can be defined without using any length-metrical unit.

14. Some Consequences

In some empirical areas the length-metrical interpretation of W as quotient Q/τ or P/μ loses its physical sense. Ratios of these forms are meaningless if neither time τ nor mass μ can associated with the movement of a quanta $A = \rho.W$. v and W are generally coupled in accordance with (13.2) and (13.1). If the length-metrical senses of W are meaningless also the length-metrical meanings x/t and p/m of v are senseless and vice versa. In these special areas the classical conceptions of v, W as ratios ('velocities') are lost.

In these areas the physical interpretation as 'movers' is still possible because v, W are generally conformal angle invariants. Also if $W = W^\wedge$ of a quantum mechanical Ψ-number cannot be interpreted as 'velocity' the interpretation as 'mover' of the quantum Ψ is possible. Also if length-metrical senses of the norm ρ of a complex Schrödinger number $\Psi = \rho.W$ are lost, a conformal physical interpretation of this norm is possible. M. Born has interpreted this norm with the help of the concept 'probability'.

In classical mechanics the concept 'velocity v' is a very fundamental one. Newton could formulate the mechanical basic equation $F = dp/dt = m.dv/dt$ only together with his definition of velocity $v := dx/dt$. This mechanical equation has lost its general physical sense because initial conditions x_0, v_0 of a movement cannot generally be realized empirically. Newton's calculus is a real calculus. His real quotient $v := dx/dt = f'(t)$ of the real function $x = f(t)$ cannot be generally used: Mathematical physics may discuss the physical meaning of quaternionic, non-real differential quotients together with the quaternionic movement of quanta $z' \to z = f(z')$ in the natural 3-dimensional space: Which term allows the interpretation of these quotients as (variable) movers W?

A microphysical movement is a movement of single quanta. A macro-physical movement is a movement of many quanta. But in both situations only geometrical/physical basic elements (quanta = quaternions) exist that can be moved. One single quantum, which is moving, can sometimes be made real extensively or intensively. If the 'intensity' $P = (m,p)$ of a quantum is realized its norm has the character of a rest mass (frequency, energy). If its 'extensity' $Q = (t,x)$ is realized its norm is a proper time.

In microphysics, if only single quanta are moved, this movement can only be grasped alternatively. A physicist may statistically prognosticate this movement but if he measures he has to decide on the dynamical or the kinematic aspect, for the intensive or the extensive feature of this microphysical movement. This creates a basis for Heisenberg's uncertainty: This postulate rests on the fact that only quanta are moving, objectively seen. In microphysics time and mass (energy) and therefore position and impulse of a single quantum cannot be fixed simultaneously.

The picture of quanta movements — 'behind' of length-metrically describable, in the classical space-time-scheme established movements — generates a new view of the duality between kinematical events (t,x) and dynamical events (m,p), formally described by special relativity[19]. This duality is the formal expression of the fact that these two length-metrical images describe alternatively either the extensive or the intensive aspect of any quantum.

Our tradition sees the space(-time)-scheme as the primary one: Material elements (e.g. mass points) are 'in' space. The quaternionic picture invites one to accept also the dual aspect: Matter (m,p) may also be seen as the primary scheme. Then the classical space-

[19] Dirac noticed and used this duality ([8], p. 110: "The theory of relativity puts energy in the same relation to time as momentum to distance"). But his theoretical background could not explain it.

scheme (t,x) is generated by the material elements. But the mutual basis — both of spatial extension and material intensity of a movement — is the purely conformal structure of quanta. Neither kinematical events (t,x) nor dynamical mass points (m,p) are primary elements of physical reality. Indeed quanta — as single things and as sets — constitute space and matter, but as conformal entities they primarily possess only angle structures. These quanta only exist as moving things. But their 'conformal' movement is a 'frozen' one because the classical all round parameter t ('time') has lost its general sense. These number-particles are 'normal' triangles with a Euclidean sum of angles (a sum of frozen movements[20]). But only these particles, not classical 'points' are primary elements of our 3-dimensional physical reality. As single, microphysical elements they cannot primarily be understood as elements "in" (length-metrical, Cartesian) space or as elements "of" (length-metrical, Cartesian) matter. Only the measurements of angles not the measurements of lengths constitute the physical description of single, spherical quanta that do not possess defined magnitudes and locations in a length-metrical world.

Can mathematical physics compose Einstein's gravitation theory in a new and more general manner if quaternionic particles displaced between individual quanta — rather than points on geodetic lines — represent physical movements? (cf. [22], Section 10, "On Pythagoras' theorem"). Can a both relativistic and quantum mechanical structures describing GENERAL MECHANICS be formulated only when quanta, not points are seen as the first and primary elements of the physical 3-dimensional space?

15. A Historical Cycle

Greek philosophers wanted to see NUMBERS as first and primary elements of reality. But Greek mathematicians could only realize 'rational numbers', ratios of natural numbers. Euclid constructed axiomatically a 'natural geometry' without knowing the concept of 'real number' and the concept of 'coordinate system'. TRIANGLES were very fundamental figures of his theory.

Newton thought of nature as embedded in 'absolute' space describable by Cartesian coordinate systems. Einstein put into perspective Newton's absolute space (and time) but always accepted local Cartesian coordinate systems (t,x).

Quaternionic skew field theory can put into perspective the kinematic and dynamic description of movements situated in 3-dimensional physical space. This physical space and its geometry can be thought primarily without of the 4-dimensional space-time concept of Minkowski and Einstein. Human beings standing in macro-physical systems of observation may use metrical concepts as 'length of time' or 'magnitude of mass (and energy)' to realize a complete Cartesian coordinate system. But to understand movements of quanta they must not always use this Cartesian concept of 'length-metrical coordinate systems'. Angles are more fundamental than lengths.

Quaternionic theory has a purely algebraic basis. But the elements of this algebra are geometrical figures — centred and directed tetraglobes, 'Euclidean TRIANGLES'. The trigonometry of these triangles is a newly written Euclidean one. The elements of this algebra constitute a 'natural geometry', free of length-metrical coordinates. Quaternionic theory sees complex quaternionic NUMBERS as first and primary geometrical elements of reality. These 'Euclidean triangles' can be perceived as elementary physical particles — first and primary elements of natural space. But these triangles can also be seen as Cartesian coordinate systems if an individual fourth point of this coordinate-tetra is fixed as a point of infinity. Physicists can fix this point but they must not do that. If they do it they possess only a very individual coordinate system, with an individual point of infinity and individual units of time and mass. Essential parts of physical reality may be describable without using length-metrical Cartesian systems. The linear part of this 'Natural Geometry', a newly written Euclidean one, may be seen as the projective ('descriptive') geometry of a - one-dimensional - NUMBER LINE[21], which possesses quaternionic points that materialize as physical quanta.

16. On Light Velocity, Action Quantum and Angle Unit

With the help of Sommerfeld's fine structure Dirac discussed the mutual status of action quantum, elementary charge and light velocity. He was convinced that one of these three basic numbers could be deduced from two of the others [9]. In using Sommerfeld's fine structure my article [22] brought the purely mathematical numbers c, h of the quaternionic calculus together with the physical basic units 'light velocity', 'action quantum' and 'elementary charge'. My point of view is related to Dirac's considerations.

[20] A dreaming scientist may feel that the three angles of a Euclidean triangle (of a tetraglobe), the three 'frozen' movements of a quantum and the three quarks of elementary particles are similar things.

[21] Mathematical tradition does not call the geometry of this line 'projective geometry' but 'Möbius geometry' if its points $\mathbf{z}$ are especially common complex numbers ($\mathbf{z} \in \mathbf{C}$). My articles [23] and [24] have developed the geometry of this number-line starting with the concept 'symmetry of conformal circles'. The geometry of the number line $\mathbf{H}$ should be developed starting with the concept 'symmetry of conformal spheres'. Likely the function of Definition 1.1, [23], defined in $\mathbf{C}$ can be generalized as a function in $\mathbf{H}$. But here a definition must pay attention to the non-commutative multiplication of $\mathbf{H}$.

What results does the quaternionic picture produce if the mutual status of action quantum and light velocity is sought; and how do these numbers behave with regard to the natural unit of angles?

Science does not possess natural units of time and mass. The units of these length-metrical magnitudes had to be chosen arbitrarily. But mathematics and physics possess a natural unit of angles. The quaternionic model leads to the concept that the physical basic unit h (or $2\pi h$) is the natural measurement of the 'full angle' (complete turn, Vollwinkel), seen geometrically. Science possesses exactly one action quantum and exactly one full angle that are realized in every quantum[22]. But the quaternionic picture with its two complementary coordinate systems demonstrates that it is natural to associate with the action quantum and therefore also with its geometrical appearance — the full angle — two natural metrical specifications, $h = i$ and $h = 1$ (or $2\pi i$ and 2π). In this way both the wave and the corpuscle aspects of physical movements can be described in one model and without contradictions.

The unit $2\pi h$ respectively the period of a circular material movement (described by this unit of angles) can also (indirectly) be used as a metrical unit of vectors (due to the correspondence of angle and vector measurement, [26], Section 4). But vectors primarily are measured by **c**. Angles essentially are periodical, vectors non-periodical magnitudes. Therefore vectors do not possess a complete natural unit. Here the constant of fine structure is empirically involved. The complete expression of the natural angle unit h does not need any empirical specification: Section 3.3 of [22] determined as the real light velocity $c/137$, as the real action quantum $h/2\pi$. 137 is an empirical, not an exact number. However 2π is an exact number.

The quaternionic picture confirmed the concept of Dirac that the physical basic numbers stand in a fundamental relationship. And the physical skew field theory can also describe this relationship mathematically. Yet quaternionic theory also makes a difference: Dirac assumed that 'in a future physical picture' elementary charge and light velocity would be the fundamental numbers and action quantum would be derivable [9].

Skew field theory says: The metrical fundamental equation $c.h = i$ describes a completely symmetrical relationship of c and h. In this regard both basic units have the same fundamental state. But h seems also a more fundamental unit because quantum action (and only this one) can be directly identified with the geometrical unit of angles.

On the other hand **c** also plays a fundamental role: The light velocity is the fundamental vectorial unit that must be identified with the two fundamental mathematical vector units **i** or **1**. Therefore again in skew field theory light velocity gets the familiar fundamental relativistic role. Above that, the two essentially different specifications of **c** by **i** and **1** are theoretically satisfying: Description of light movement with the help of Maxwell's equations is a suitable description if $\mathbf{c}^2 > 0$. Maxwell's equations are not responsible if $\mathbf{c}^2 < 0$. In this state the movement of matter has to be described. Movements of matter produce light and movements of matter are induced by light. In this states h not **c** is a real number. These states bring into play quanta of light of the form $h\nu$.

17. Summary

Finally three essential characteristic features of the quaternionic picture will be underlined:

- With the help of this picture the relativistic 4-dimensional space-time-scheme again can be reduced to a model in three dimensions. The movement of particles — with their complementary structures — can be immediately perceived as the movement of numbers. 'Numbers', 'quaternions', 'spherical particles', 'quanta' appear as synonymous. The geometrical fundament of the 3 dimensional mechanical space possesses a conformal, purely angle-metrical structure. The wave aspect of quanta can be perceived without the use of length-metrical features.

- The basic physical units action quantum h and light velocity c do not only appear as units of an empirical origin. They already are purely mathematical constituents of the quaternionic triangle model, bound by the metrical fundamental equation $i = h.c$. The mathematical scalar h that is physically the action quantum, measures geometrically the sum of angles realized in every quaternion.

- The vectorial quaternionic unit **i** is the product of light velocity and action quantum, $\mathbf{i} = h.\mathbf{c}$. In a precise sense this 'imaginary' quaternion is more real than the basic metrical numbers **c** and h: It depends on the chosen coordinates of **c** and h whether the corpuscular or the wave aspect of **i** is realized empirically. The number **i** itself exists independently of its complementary coordinates as a geometrical figure (i.e. as orthogonal spherical particle, trigonometrical determination of the right angle, prototype of a Cartesian system). — The question is how this 'absolute quaternionic unit' appears experimentally[23] in microphysics.

[22] A quantum is a tetraglobe. In each point of this (Euclidean) triangle the three triangle angles appear in a twofold manner, as angle and as apex angle. In any of the four tetraglobe-points the sum of these 2×3 angles is $2\pi h$, the double sum of the angles in a Euclidean triangle.

[23] A conformal circle (2^0-circle) does not possess an extent (a radius). But every 2^k-circle ($k = 0, \ldots, 3$) ('polycircle') pos-

And lastly with the help of ten headlines I will collect the reasons in favour of my identification of numbers and quanta:

1. *Points are problematic models.* Theoretical physics is looking for a geometrical and physical basic concept that can substitute the traditional model "point". A "more general point" is quested, which possesses an inner structure. The word "string" may indicate other attempts going in the same direction. Numbers, realized as conformal triangles, possess an inner structure of that kind.

2. *Dualism of particles and waves.* A complex quaternionic number, which possess the shape of a trigon, can be perceived as an element that possesses — in a more general conformal form — both the structure of a classical "point" (a corpuscular element of moving) and the structure of a "plane" (the element of a wave).

3. *Three-dimensional natural space.* Physical quanta are elementary natural elements existing in 3-dimensional space of our experience. With the help of the triangle-picture also numbers can be seen as elementary geometrical and physical figures in the space of our practical knowledge. Mathematics has proved that the original number concept cannot be transferred into spaces with higher dimensions. If numbers are accepted as primary and first elements of natural space, the fact of three dimensions, which characterized the physical space, is understood theoretically. Other initial stages to substitute the traditional space element "point" by elements with inner structure are very busy to reduce many formal dimensions to the natural three one. These natural three dimensions are a pure mathematic fact if physical quanta are mathematic numbers.

4. *The question of distances and locality.* Quantum physics has exposed that the concept "length" ("diameter of an atom"), and also the concept "location" (for instance of an electron) often lose their sense in microphysics. Like a microphysical quantum also a number that is seen as a conformal, geometrical sphere, generally has not a defined diameter. Only together with special conditions we can "locate" a (physical) quanta and a (mathematical) number. For example mathematics can only do that if a right-angled trigon is perceived as a Cartesian system, if in this system the existence of a length unit is presupposed and if a point-location is described in using numbers as pointers.

5. *Quanta are triangles.* Quaternions cannot only be seen as vectors (points) but also as conformal triangles. This double picture of numbers illustrates geometrically the double structure of physical quanta. A physical movement cannot exclusively be described by a vector function $t \to x = x(t)$ of time t (Heisenberg). But a quaternion can exclusively be seen as the element of a physical movement in our natural space. Quaternionic quanta can always be written as a sum of real and vector parts. But these additive parts cannot always be seen (measured) as a real time parameter t and a vectorial location parameter x. In microphysics the triangle product representation $A = \rho.W$ of a quanta is a more fundamental and a more essential one than the vector sum representation $A = \alpha + a$. Quanta of this product form are Schrödinger's Ψ-numbers. Physical movements can still be described by Ψ-functions, if the classical vector representation of movements has lost its physical sense.

6. *Mechanics as Geometry — Geometry as Mechanics.* Using the concept "number line" mathematics started with a geometric view of (real) numbers. Gauß (together with Wessel, Buée, Argand) extrapolated this geometrical view to complex numbers. A physical but quaternionic mechanical field theory is based on a complete geometric view of (commutative and noncommutative complex) numbers. Numbers are not only vectors (points) but also conformal triangles. — A complete geometric field theory of numbers may also complete the algebraic and function theoretical/differential geometric view of mechanics that was prepared by Descartes, founded in a classical form by Newton and continued by Riemann and Einstein. Can Einsteinean "points" on geodetic lines be substituted by a differential geometric connection of triangle quanta between individual — but not always local — spaces?

7. *A new angle concept.* The conformal picture of numbers uncovered that Euclidean trigonometry of (individual, local) physical spaces can be developed without using concepts as "length" and "straight line". Angles can be defined without using lengths and straight lines [23,24]. The

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sesses the conformal invariant 'geometrical charge' $\varepsilon = -1$ ("charge" in [25], Def. 5.1. Cf. also Remark 1 in Section 2 of [23]). If the spin concept in the sense of [25], Def. 5.2, is transferred to all known polycircles so all these figures, not only the quaternionic 2^2-circles have the type of physical fermions (spin direction **i**, basic spin $^1/_2$). ε and the Euclidean sum of angles $h\pi$ are two measures of the same geometrical/physical entity. The question whether a certain physical elementary particle can be identified with the quaternionic particle **i**, and if the 24 known elementary spin-$^1/_2$-particles can be perceived as angles, quaternions, 2^k-circles is not a topic of this article. A perhaps possible connection of geometrical charge and physical elementary charge (electronic charge) must also be discussed separately.

concept "angle" is more fundamental than the concept "length". Einstein's relativity leads to a reflection of (length-metrical!) basic concepts as "time", "distance" and "mass"; relativistic theory could identify mass and energy. A theoretical identification of numbers and quanta leads to a reflection of the physical meaning of angles. This reflection uncovers that only a natural unit of angles but not natural units of time and mass do exist. Good reasons lead to an identification of action quantum and angle unit.

8. *'Angles = Quarks' is a heuristic point of view.* Systematics of elementary particles was initiated by using the quark concept. Thereby, the fundamental conception was to see a particle as a system of three quarks. But quarks cannot be seen independently. This is a typical — and also mysterious — behave of quarks. Numbers, geometrical systems of a spherical shape, are triangles characterized by the metrical numbers of three angles. "Angle" is a fundamental concept of number-geometry. "Angle" is more fundamental than "distance". Angles appear physically as more general forms of movements ("frozen movements"). But angles obviously — not mysteriously — cannot be seen independently (of figures); they can only be seen as parts (constituents) of (conformal circle) figures.

9. *Uniqueness of the numerical skew field* . Mathematics possesses one and only one number skew field. Only the infinitesimal calculus of the quaternionic skew field is strictly comparable with the calculus of real and common complex fields: Only these three fields — real numbers, common complex numbers and quaternionic numbers — are topological fields that are connected and locally compact (Frobenius, Pontrjagin [21]). As elements of this "specially honoured" field structure numbers/quanta have to be seen as aboriginal inhabitants of the pure mathematical world. Also science studies only one natural world. In this sense both the physical and the mathematical world possess one and only one skew field; and (physical) quanta as well as (mathematic) numbers possess a complementary structure.

10. *Physical movements are quaternionic functions.* Classical mechanics is based on the axiom: A physical movement is either a wave or a point movement. Quantum mechanics has accepted the new axiom: A physical movement has to be seen both as a wave movement and as a point movement. The perfect usage of quaternionic numbers for describing mechanics is traditionally hindered by the axiom: Multiplication of complex numbers must geometrically be perceived with the help of rotations[24] — and Lorentz transformations cannot be seen as (real) rotations.

    But this contradiction can be cleared if the trigonometry of quaternionic numbers accepts both real and imaginary metrical numbers of angles; and if the classical, Euclidean differences of straight lines and Euclidean circles are eliminated in using the concept "conformal circle" (Remarks 5–7, Section 2 of [23]). The first element of every physical movement ("Einstein-transformation") can be seen both as a rotation and a Lorentz transformation if the classical picturing of number multiplication by the composition of rotations is substituted by the composition of triangles. Quaternionic triangles are both elements of a point movement and a wave movement; quaternionic quanta are both particles and waves.

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